

Acoustic geometry obtained through the perturbation of the Bernoulli's constant

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Abstract

For accretion onto astrophysical black holes, we demonstrate that linear perturbation of Bernoulli's constant defined for inviscid ir-rotational adiabatic flow of perfect ideal fluid gives rise to phenomena of analogue gravity. The formulation of our work is done in Newtonian framework and also within the General relativistic frame work, i.e; considering a static space time back ground, as well. The resulting structure of the analogue acoustic metric is similar to the acoustic metric found in perturbing velocity potential and mass accretion rate. Stability analysis of the background stationary flow is performed by studying the equation of the massless scalar field generated by the perturbation of Bernoulli's constant.

1 Introduction

Unruh's[1] pioneering work shows that Linear perturbation of velocity potential defined for ir-rotational inviscid flow of fluid astoundingly gives rise to phenomena of analogue gravity. corresponding to the propagation of such perturbation obeys the Klein Gordon equation for a massless scalar field in curved space time. If the flow is of converging nature, spherically symmetric and at some radius the bulk velocity of the flow crosses the sound speed then one can find acoustic black holes analogous to the Schwarzschild black holes. Visser's work[2] deals with some details in the topic considering several interesting geometry like vortex geometry and others. The basic idea is that the background flow is considered to be steady and linear perturbation having both explicit time dependence and space variation in general is introduced in the flow, i.e; the density, velocity are perturbed linearly accordingly but the source term (independent of flow of the fluid, i.e; independent of fluid velocity, density etc) in the Euler equation is not disturbed.

Astrophysical accretion is a natural phenomena where linear perturbation of flow can be considered. It is assumed that the accretion to be non-self gravitating so that the source term in Euler equation remains unchanged. Specifically for transonic accretion, the accretion flow is a suitable candidate to produce black hole horizon like effects, i.e; there is no way for sound to travel from supersonic flow region towards subsonic region of stationary background solution. As for example, in case of Bondi accretion[3], the flow is inviscid and ir-rotational as well. Sub-Keplarian disk accretions[4, 5] at the center of our galaxy are also good candidates. Mass accretion rate is a quantity having a reasonable physical significance in accretion phenomena. Linear perturbation of mass accretion rate in sub-Keplarian disk accretion in non relativistic framework also behaves like massless scalar field in curved space-time[6], i.e; analogue gravity also emerges when accretion rate is perturbed.

Several works have been done in General relativistic framework as well. Linear perturbation of velocity potential in curved space-time background shows analogue gravity effect[7]. Similarly linear perturbation of mass accretion rate in accretion of perfection fluid in curved space-time background also shows same

effect[8, 9, 7].

In this work, we've shown that linear perturbation of another quantity, the Bernoulli's constant which is integral solution of the corresponding Euler equation, also produces similar acoustic geometry. The whole work is being done in non-relativistic framework and relativistic framework as well. Accretion phenomena of adiabatic flow is chosen to illustrate the fact. Radial accretion having spherical symmetry as well as disk accretion having axial symmetry are considered.

2 Acoustic Gravity in non-relativistic framework

In non-relativistic frame work, fluid velocity is much less than light speed. The momentum conservation equations and mass conservation equation for fluid is taken[10] according to Newton's laws of dynamics. The continuity equation of fluid is given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad (1)$$

where ρ, \vec{v} are fluid density and velocity respectively. Euler momentum equation for inviscid flow in an external field in general is given by

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\nabla \psi - \frac{\vec{\nabla} p}{\rho} \quad (2)$$

where the potential function of the external conservative field is ψ and pressure at any point of the fluid is p.

The flow is taken to be irrotational.

$$\vec{\nabla} \times \vec{v} = 0 \quad (3)$$

Using irrotationality condition one can write Euler equation as

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla} \left(\frac{1}{2} \vec{v}^2 + \int \frac{dp}{\rho} + \psi \right) = 0 \quad (4)$$

Bernoulli's constant, ζ is given by

$$\zeta = \frac{1}{2} \vec{v}^2 + \int \frac{dp}{\rho} + \psi \quad (5)$$

For steady irrotational flow it's a constant along the streamline. Adiabatic sound speed is given by

$$c_s^2 = \frac{dp}{d\rho} = \frac{\gamma p}{\rho} \quad (6)$$

We assume there is a stationary solution in general for the above equations, and we introduce linear perturbations in the fluid discussed in the next section.

2.1 General procedure to obtain the acoustic metric

Linear perturbation of fluid velocity and fluid pressure is introduced as

$$\vec{v}(\vec{r}, t) = \vec{v}_0(\vec{r}) + \vec{v}'(\vec{r}, t)$$

$$\rho(\vec{r}, t) = \rho_0(\vec{r}) + \rho'(\vec{r}, t)$$

where $\rho_0(\vec{r}), \vec{v}_0(\vec{r})$ are the stationary solution of fluid density and velocity field and $\vec{v}'(\vec{r}, t), \rho'(\vec{r}, t)$ are the introduced linear perturbation terms in the velocity and the density of the fluid.

As a result, the linear perturbation term in Bernoulli's constant is given by

$$\zeta' = \vec{v}_0 \cdot \vec{v}' + \frac{c_{s0}^2}{\rho_0} \rho' \quad (7)$$

c_{s0} is the stationary unperturbed sound speed. Continuity equation in terms of linear perturbation is given by

$$\frac{\partial \rho'}{\partial t} + \vec{\nabla} \cdot (\rho' \vec{v}_0 + \rho_0 \vec{v}') = 0 \quad (8)$$

Momentum equation in terms of linear perturbation is given by

$$\frac{\partial \vec{v}'}{\partial t} + \vec{\nabla}(\zeta') = 0 \quad (9)$$

We have used equation (7) to find the above equation. Using equation (7) and equation (9) and taking another partial time derivative in equation (8) we get

$$\partial_\mu (f^{\mu\nu}(\vec{r}) \partial_\nu) \zeta'(\vec{r}, t) = 0 \quad (10)$$

where $f^{\mu\nu}(\vec{r})$ in Cartesian coordinate is given by

$$f^{\mu\nu}(\vec{r}) = \frac{\rho_0}{c_{s0}^2} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \dots & \dots & \dots \\ -v_0^j & \vdots & c_{s0}^2 \delta^{ij} - v_0^i v_0^j \end{bmatrix} \quad (11)$$

where i, j run over 1,2,3 representing three spatial dimensions. This $f^{\mu\nu}$ is exactly the same as $f^{\mu\nu}$ obtained when velocity potential is perturbed[2]

Massless scalar field equation in a space-time background is given by

$$\square \varphi = \frac{1}{\sqrt{-g}} (\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu) \varphi = 0 \quad (12)$$

where φ is the scalar field, $g^{\mu\nu}$ is the background metric and g is the determinant of the metric. Comparing equation (12) and equation (10)

$$f^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \quad (13)$$

Immediately one can get

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4 g^{-1} = g = -\frac{\rho_0^4}{c_{s0}^2} \quad (14)$$

So $g^{\mu\nu}$ is given by

$$g^{\mu\nu}(\vec{r}) = \frac{1}{\rho_0 c_{s0}} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \dots & \dots & \dots \\ -v_0^j & \vdots & c_{s0}^2 \delta^{ij} - v_0^i v_0^j \end{bmatrix} \quad (15)$$

The acoustic metric is

$$g_{\mu\nu}(\vec{r}) = \frac{\rho_0}{c_{s0}} \begin{bmatrix} -(c_{s0}^2 - v_0^2) & \vdots & -v_0^j \\ \dots & \dots & \dots \\ -v_0^j & \vdots & \delta_{ij} \end{bmatrix} \quad (16)$$

Acoustic metric interval can be expressed as

$$ds^2 = \frac{\rho_0}{c_{s0}} [-(c_{s0}^2 - v_0^2) dt^2 - 2 dt v_0^i d\vec{x} + d\vec{x}^2] \quad (17)$$

The same kind of analysis can be done for isothermal flow as well and the metric will be same except that the definition of sound speed will be different there, in the acoustic metric, sound speed will be appearing as a constant rather than a function of position vector.

The metric appearing in equation (16) has 3+1 dimension. It reduces to 1+1 dimension when symmetries in the flow is considered. The next section deals with some astrophysical accretion phenomenon having different kind of symmetries.

2.2 Spherically Symmetric Radial flow

Bondi accretion[3] is spherically symmetric and radial. The continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho v r^2) = 0 \quad (18)$$

Euler momentum equation is given by

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\frac{GM}{r^2} - \frac{1}{\rho} \frac{\partial p}{\partial r} \quad (19)$$

where M is the mass of the star and G is gravitational constant. Bernoulli's constant is given by

$$\zeta = \frac{1}{2}v^2 + \int \frac{dp}{\rho} - \frac{GM}{r} \quad (20)$$

Introducing linear perturbation in adiabatic flow

$$v(r, t) = v_0(r) + v(r, t)'$$

$$\rho(r, t) = \rho_0(r) + \rho(r, t)'$$

Perturbation in Bernoulli's constant is given by

$$\zeta' = v_0 v' + \frac{c_{s0}^2}{\rho_0} \rho' \quad (21)$$

Now in the same way discussed earlier we find that the linear perturbation of Bernoulli's constant obeys massless scalar field equation in acoustic analogue of space-time background.

$$\partial_\mu (f^{\mu\nu}(r) \partial_\nu) \zeta'(r, t) = 0 \quad (22)$$

where

$$f^{\mu\nu}(r) = \frac{\rho_0 r^2}{c_{s0}^2} \begin{bmatrix} -1 & -v_0 \\ -v_0 & c_{s0}^2 - v_0^2 \end{bmatrix} \quad (23)$$

$f_{\mu\nu}$ is taken as effective metric[6]. Hence 2×2 effective acoustic metric is given by

$$g_{\mu\nu}^{eff}(r) = \frac{1}{\rho_0 r^2} \begin{bmatrix} -(c_{s0}^2 - v_0^2) & -v_0 \\ -v_0 & 1 \end{bmatrix} \quad (24)$$

Observation of the acoustic metric shows that acoustic horizon is produced.

Similarly, the same analysis can be done for isothermal flow as well.

2.3 Axially symmetric sub-Keplarian Disk Geometries

Low angular momentum axisymmetric blackhole accretion[4, 5] is a good candidate where analogue gravity emerges too[6]. We don't need to consider viscosity in such weakly rotating sub-Keplarian flows due to low angular momentum of the infalling fluid. There are mainly three disk models for sub-Keplarian disk by categorising them with respect to disk height or thickness H . H is taken to be constant in the simplest possible model, i.e; in uniform thickness disk model. In conical model[4] the disk thickness H is proportional to cylindrical radial distance from the accretor. In the most physical disk model, i.e; in Vertical equilibrium model[5, 11], the disk height $H(r)$ is a function of cylindrical radial distance r from the accretor such that there is no flow along z direction considering the equatorial plane of the disk to be on the $X - Y$ plane. In this type of models, due to symmetry, the problem becomes effectively 1+1 dimensional. In the next section we consider the non-trivial disk model, i.e; vertical equilibrium disk model.

2.4 Vertical Equilibrium Disk Accretion

Continuity equation in cylindrical polar coordinate in disk accretion having axial symmetry and having no net flow in z direction is given by

$$\frac{\partial \bar{\rho}(r, z)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\bar{\rho}(r, z) \bar{v}(r, z) r) = 0 \quad (25)$$

where $\bar{\rho}(r, z)$ and $\bar{v}(r, z)$ are the fluid density and radial velocity at a cylindrical radial distance r and at height z from the equatorial plane of the disk. Now averaging in z direction over disk height H

$$\frac{\partial \rho(r)}{\partial t} + \frac{1}{rH(r)} \frac{\partial}{\partial r} (\rho(r) v(r) r H(r)) = 0 \quad (26)$$

where $\rho(r)$ and $\rho(r)v(r)$ are the averaged fluid density and momentum respectively. The problem is now reduced in 1+1 dimension. Euler momentum equation is given by

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = -\psi'(r) - \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\lambda^2}{r^3} \quad (27)$$

where $\psi'(r)$ is the external field term and in this case it is gravitational force per unit mass of fluid exerted by the accretor. λ is the angular momentum of the fluid having small value. Bernoulli's constant is given by

$$\zeta = \frac{1}{2} v^2 + \int \frac{dp}{\rho} + \psi(r) + \frac{\lambda^2}{2r^2} \quad (28)$$

Considering thin disk in vertical equilibrium and adiabatic flow, balancing pressure gradient force term and gravitational force term along z direction, vertical equilibrium condition is given[6, 11]

$$H(r) = c_s(r) \sqrt{\frac{r}{\gamma \psi'}} \quad (29)$$

As a consequence, continuity equation is given by

$$\partial_t (\rho^{\frac{\gamma+1}{2}}) + \frac{\sqrt{\psi'}}{r^{\frac{3}{2}}} \partial_r \left(\frac{\rho^{\frac{\gamma+1}{2}} v r^{\frac{3}{2}}}{\sqrt{\psi'}} \right) \quad (30)$$

Introducing linear perturbation in the adiabatic flow, perturbation of Bernoulli's constant is given by

$$\zeta'(r, t) = v_0 v' + \frac{c_{s0}^2 \sigma}{\rho_0^{\frac{\gamma+1}{2}}} \delta(\rho^{\frac{\gamma+1}{2}}) \quad (31)$$

where $\sigma = \frac{2}{\gamma+1}$ and $\delta(\rho^{\frac{\gamma+1}{2}})$ is linear perturbation in $\rho^{\frac{\gamma+1}{2}}$.

Linear perturbation of Bernoulli's constant obeys massless scalar wave equation

$$\partial_\mu (f^{\mu\nu}(r) \partial_\nu) \zeta'(r, t) = 0 \quad (32)$$

where

$$f^{\mu\nu}(r) = \frac{\rho_0^{\frac{\gamma+1}{2}} r^{\frac{3}{2}}}{c_{s0}^2 \sigma \sqrt{\psi'}} \begin{bmatrix} -1 & -v_0 \\ -v_0 & \sigma c_{s0}^2 - v_0^2 \end{bmatrix} \quad (33)$$

$f_{\mu\nu}$ is taken as effective metric. Hence 2×2 effective acoustic metric is given by

$$g_{\mu\nu}^{eff}(r) = \frac{\sqrt{\psi'}}{\rho_0^{\frac{\gamma+1}{2}} r^{\frac{3}{2}}} \begin{bmatrix} -(\sigma c_{s0}^2 - v_0^2) & -v_0 \\ -v_0 & 1 \end{bmatrix} \quad (34)$$

For isothermal flow one similarly gets acoustic metric like equation (34) where γ is 1 and sound speed is a constant number.

2.5 Constant Height Disk Accretion

In case of constant thickness model, H is a constant. The linear perturbation of Bernoulli's constant is given by

$$\zeta'(r, t) = v_0 v' + \frac{c_{s0}^2}{\rho_0} \rho' \quad (35)$$

The linear perturbation of continuity equation is given by

$$\partial_t(\rho') + \frac{1}{rH} \partial_r(H(\rho' v_0 + \rho_0 v')) = 0 \quad (36)$$

where H is a non zero constant number. The linear perturbation of momentum equation is given by

$$\partial_t(v') + \partial_r(\zeta') = 0 \quad (37)$$

Now proceeding in the same way discussed earlier one gets equation of massless scalar field in curved space time background

$$\partial_\mu f^{\mu\nu}(r) \partial_\nu \zeta' = 0 \quad (38)$$

where after taking inverse of $f^{\mu\nu}(r), f_{\mu\nu}(r)$ can be taken as 2×2 effective metric as

$$f_{\mu\nu} = g_{\mu\nu}^{eff}(r) = \frac{1}{\rho_0 r H} \begin{bmatrix} -(c_{s0}^2 - v_0^2) & -v_0 \\ -v_0 & 1 \end{bmatrix} \quad (39)$$

For conical disk model $H \propto r$. Just like the constant height disk model, linear perturbation in fluid does not have any influence on disk height. Hence the procedure of getting massless Klein Gordon equation is exactly same and the effective acoustic metric is exactly same as obtained in constant height disk model.

3 Acoustic Gravity in curved space-time background

In the present work we consider the following metric for static spacetime

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \quad (40)$$

where the metric elements are functions of r and can also be functions of θ and ϕ . We assume a perfect fluid with the energy-momentum tensor given by

$$T^{\mu\nu} = (\epsilon + p)v^\mu v^\nu + p g^{\mu\nu} \quad (41)$$

with the velocity four-vector normalized as $v^\mu v_\mu = -1$. The fluid is assumed to be ideal and so obeys equation of state for ideal gas. Also it is assumed to be under adiabatic condition i.e it obeys barotropic equation of state i.e $p = k\rho^\gamma$. The specific enthalpy of the fluid is given by

$$h = \frac{\epsilon + p}{\rho} \quad (42)$$

The speed of sound for adiabatic flow is given by

$$c_s^2 = \frac{\partial p}{\partial \epsilon} \quad (43)$$

which can be also written as[7]

$$c_s^2 = \frac{\rho}{h} \frac{\partial h}{\partial \rho} \quad (44)$$

In our calculation of acoustic geometry we make use of two basic equations. First one is the continuity equation given by

$$\nabla_\mu(\rho v^\mu) = 0 \quad (45)$$

and the second one is the irrotationality condition as the fluid is assumed to be irrotational. The condition is given by

$$\partial_\mu(hv_\nu) - \partial_\nu(hv_\mu) = 0 \quad (46)$$

3.1 Spherically Symmetric Radial Flow

In the first case in curved space-time background we derive the acoustic geometry for spherically symmetric flow. This implies that $v_\theta = v_\phi = 0$ and all the derivatives with respect to θ and ϕ vanish. Using $\mu = t$ and $\nu = r$ in the irrotationality condition equation given by equation (46) gives

$$\partial_t(hv_r) - \partial_r(hv_t) = 0 \quad (47)$$

In stationary case where ∂_t term vanishes the above equation implies $\partial_r(hv_t) = 0$. So for stationary flow $\zeta = hv_t$ is a constant of the flow. This is called the specific energy for adiabatic flow or the Bernoulli's constant. The continuity equation given by equation (45) becomes

$$\frac{1}{\sqrt{-g}}\partial_t(\sqrt{-g}\rho v^t) + \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}\rho v) = 0 \quad (48)$$

where $v = v^r$ is the radial velocity. Using the normalization condition of the four-velocity given $v^\mu v_\mu = -1$, v^t can be expressed as

$$v^t = \sqrt{\frac{1 + g_{rr}v^2}{g_{tt}}} \quad (49)$$

Now we linearly perturb the radial velocity, density and the Bernoulli's constant about their stationary values.

$$v(r, t) = v_0(r) + v'(r, t) \quad (50)$$

$$\rho(r, t) = \rho_0(r) + \rho'(r, t) \quad (51)$$

and

$$\zeta(r, t) = \zeta_0 + \zeta'(r, t) \quad (52)$$

Using these quantities we do linear perturbation of the continuity equation and the irrotationality condition equation given by equation (48) and equation (47) respectively.

Linear perturbation of the irrotationality condition equation gives the following equation

$$\partial_r \zeta' = g_{rr} h_0 \partial_t v' + \frac{g_{rr} h_0 v_0 c_{s0}^2}{\rho_0} \partial_t \rho' \quad (53)$$

where h_0 is the stationary or background value of the enthalpy h and $c_{s0}^2 = \frac{\rho_0}{h_0} \frac{\partial h}{\partial \rho}$ again perturbing $\zeta = hv_t = -g_{tt}hv^t$ gives the equation

$$\zeta' = -g_{tt} h_0 \alpha v' - \frac{g_{tt} v_0^t h_0 c_{s0}^2}{\rho_0} \rho' \quad (54)$$

where $\alpha = \frac{g_{rr} v_0}{g_{tt} v_0^t}$ and we have used the normalization condition of four-velocity to obtain $(v^t)' = \alpha v'$. Taking time derivative of the above equation gives

$$\partial_t \zeta' = -g_{tt} h_0 \alpha \partial_t v' - \frac{g_{tt} v_0^t h_0 c_{s0}^2}{\rho_0} \partial_t \rho' \quad (55)$$

Using equation (53) and equation (55) we are able to write $\partial_t v'$ and $\partial_t \rho'$ in terms of ζ' only. Thus we have

$$\partial_t v' = \frac{-1}{\Delta} \left[\frac{g_{rr} h_0 v_0 c_{s0}^2}{\rho_0} \partial_t \zeta' + \frac{g_{tt} v_0^t h_0 c_{s0}^2}{\rho_0} \partial_r \zeta' \right] \quad (56)$$

$$\partial_t \rho' = \frac{1}{\Delta} [g_{rr} h_0 \partial_t \zeta' + g_{tt} h_0 \alpha \partial_r \zeta'] \quad (57)$$

where $\Delta = -\frac{g_{rr} h_0^2 c_{s0}^2}{\rho_0 v_0^t}$

Linear perturbation of the continuity equation gives

$$\rho_0 \alpha \partial_t v' + v_0^t \partial_t \rho' + \frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} \rho_0 v' + \sqrt{-g} v_0 \rho') = 0 \quad (58)$$

taking the time derivative of the above equation gives

$$\partial_t(\sqrt{-g}\rho_0\alpha\partial_tv') + \partial_t(\sqrt{-g}v_0^t\partial_t\rho) + \partial_r(\sqrt{-g}\rho_0\partial_tv') + \partial_r(\sqrt{-g}v_0\partial_t\rho') = 0 \quad (59)$$

Substituting ∂_tv' and $\partial_t\rho'$ in the above equation using equation (56) and equation (57) gives

$$\begin{aligned} & \partial_t\left[\frac{\sqrt{-g}g_{rr}h_0}{\Delta v_0^t}\left\{\frac{g_{tt}(v_0^t)^2(1-c_{s0}^2)+c_{s0}^2}{g_{tt}}\right\}\partial_t\zeta'\right] + \partial_t\left[\frac{\sqrt{-g}h_0g_{rr}v_0}{\Delta}\{1-c_{s0}^2\}\partial_r\zeta'\right] \\ & + \partial_r\left[\frac{\sqrt{-g}h_0g_{rr}v_0}{\Delta}\{1-c_{s0}^2\}\partial_t\zeta'\right] + \partial_r\left[\frac{\sqrt{-g}g_{rr}h_0}{\Delta v_0^t}\left\{\frac{v_0^2g_{rr}(1-c_{s0}^2)-c_{s0}^2}{g_{rr}}\right\}\partial_r\zeta'\right] = 0 \end{aligned} \quad (60)$$

The above equation is of the form $\partial_\mu(f^{\mu\nu}\partial_\nu\zeta') = 0$ with $f^{\mu\nu}$ given by after multiplying by -1

$$f^{\mu\nu} = \frac{\sqrt{-g}\rho_0}{h_0} \begin{bmatrix} -g^{tt} + (v_0^t)^2(1 - \frac{1}{c_{s0}^2}) & v_0v_0^t(1 - \frac{1}{c_{s0}^2}) \\ v_0v_0^t(1 - \frac{1}{c_{s0}^2}) & g^{rr} + v_0^2(1 - \frac{1}{c_{s0}^2}) \end{bmatrix} \quad (61)$$

3.2 Axially symmetric disk flow

Three low angular momentum disk models (as discussed in previous sections) are considered for adiabatic flow. The normalization condition is given by

$$v_\mu v^\mu = -1 \quad (62)$$

The spherically symmetric diagonal metric of equation (40) is considered. We consider the dynamics only on the equatorial plane ($\theta = \frac{\pi}{2}$) plane of the disk. The accretion flow is irrotational, i.e; it obeys equation (46). The infalling fluid has a small azimuthal component of velocity, v^ϕ . From equation (62)

$$(v^t)^2 = \frac{1 + g_{rr}(r)v^2 + g_{\phi\phi}(r)(v^\phi)^2}{g_{tt}(r)} \quad (63)$$

Similarly, equation (47) gives Bernoulli's constant. Using equation equation (46) and axial symmetry of the flow

$$\partial_t(hv_\phi) = 0$$

$$\partial_r(hv_\phi) = 0$$

$$\Rightarrow hv_\phi = \text{constant} = \ell \quad (64)$$

hv_ϕ is called specific angular momentum and it is a constant number for non-stationary flow as well due to irrotationality and azimuthal symmetry. We assume that there is a stationary solution of the accretion($v_0(r)$, $\rho_0(r)$, $v_0^\phi(r)$, ζ_0) and linear perturbation is introduced.

$$v(r, t) = v_0(r) + v'(r, t) \quad (65)$$

$$\rho(r, t) = \rho_0(r) + \rho'(r, t) \quad (66)$$

$$v^\phi(r, t) = v_0^\phi(r) + v'_\phi(r, t) \quad (67)$$

and

$$\zeta(r, t) = \zeta_0 + \zeta'(r, t) \quad (68)$$

The symbols carries usual meaning as before. The addition of linear perturbations do not make the accretion flow to violate irrotationlity, azimuthal symmetry (obvious from the abve expressions). The accretion flow is still inviscid and adiabatic. From equation (64), linear perturbation term, ℓ' is given by

$$\ell' = 0 \quad (69)$$

Using equation (44) and equation (69) we get

$$v'_\phi = -\frac{v_0^\phi c_{s0}^2}{\rho_0} \rho' \quad (70)$$

Using equation (63) and equation (70) we get

$$(v^t)' = \alpha_1(r)v' + \alpha_2(r)\rho' \quad (71)$$

where

$$\begin{aligned} \alpha_1(r) &= \frac{g_{rr}v_0}{g_{tt}v_0^t} \\ \alpha_2(r) &= -\frac{g_{\phi\phi}(v_0^\phi)^2 c_{s0}^2}{g_{tt}v_0^t \rho_0} \end{aligned}$$

Irrotationality condition gives

$$\partial_r \zeta' = f_1(r)\partial_t \rho' - f_2(r)\partial_t v' \quad (72)$$

where

$$\begin{aligned} f_1(r) &= \frac{g_{rr}v_0 h_0 c_{s0}^2}{\rho_0} \\ f_2(r) &= -g_{rr}h_0 \end{aligned}$$

Using equation (71) and expression of $\zeta (= -h g_{tt} v^t)$

$$\partial_t \zeta' = -f_3(r)\partial_t \rho' + f_4(r)\partial_t v' \quad (73)$$

where

$$\begin{aligned} f_3(r) &= \frac{g_{tt}v_0^t h_0 c_{s0}^2}{\rho_0} + g_{tt}h_0 \alpha_2 \\ f_4(r) &= -g_{tt}h_0 \alpha_1 \end{aligned}$$

From equation (72) and (73) we get

$$\partial_t \rho' = g_4(r)\partial_r \zeta' + g_2(r)\partial_t \zeta' \quad (74)$$

$$\partial_t v' = g_3(r)\partial_r \zeta' + g_1(r)\partial_t \zeta' \quad (75)$$

where

$$\begin{aligned} g_i &= \frac{f_i}{\Delta} \quad \text{where } i \in \mathbb{N} \text{ and } i = 1 \text{ to } 4 \text{ where } \mathbb{N} \text{ is the set of natural numbers.} \\ \text{and } \Delta &= f_1 f_4 - f_2 f_3 = \frac{g_{rr} h_0^2 c_{s0}^2}{\rho_0 v_0^t} \end{aligned}$$

Until and unless we don't have the expression of disk height, we can not use the continuity equation. In the next section several sub-Keplarian (discussed before) disk models are considered.

3.2.1 Vertical Equilibrium Disk Model

The expression of $H(r)$ ¹ satisfying vertical equilibrium condition is given by [12, 9]

$$H(r)^2 v_\phi^2 F(r) = \frac{p}{\rho}$$

¹ $H(r)$ is not the flow thickness of the disk, this is a dimensionless quantity which appears in continuity equation after averaging in θ direction.

The linear perturbation of H is H' and stationary solution of H is $H_0(r)$. Using barotropic equation and equation (70)

$$\frac{\partial_t H'}{H_0} = \frac{\beta}{\rho_0} \partial_t \rho' \quad (76)$$

where $\beta = c_{s0}^2 + \frac{\gamma-1}{2}$.

Continuity equation is given by

$$\frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} \rho v^t H) + \frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} \rho v H) = 0 \quad (77)$$

Introducing linear perturbation in the fluid and using equation (71) and (76), one gets after partially differentiating equation (77),

$$(\partial_t F_1 \partial_t) \rho' + (\partial_t F_2 \partial_t) v' + (\partial_r F_3 \partial_t) \rho' + (\partial_r F_4 \partial_t) v' = 0 \quad (78)$$

where each F_i ($i \in \mathbb{N}, i = 1$ to 4) is function of r , the expressions are given below

$$\begin{aligned} F_1(r) &= \sqrt{-g} (H_0 v_0^t (1 + \beta) + \alpha_2 \rho_0 H_0) \\ F_2(r) &= \sqrt{-g} \rho_0 H_0 \alpha_1 \\ F_3(r) &= \sqrt{-g} H_0 v_0 (1 + \beta) \\ F_4(r) &= \sqrt{-g} \rho_0 H_0 \end{aligned}$$

Using equation (74), (75) and (78) we get

$$\partial_\mu (f^{\mu\nu}(r) \partial_\nu) \zeta' = 0 \quad (79)$$

where μ, ν indices run over t and r . 2×2 matrix, $f^{\mu\nu}$ is given by

$$f^{\mu\nu} = \frac{\sqrt{-g} H \rho_0}{h_0} \begin{bmatrix} -g^{tt} + (v_0^t)^2 (1 - \frac{1+\beta}{c_{s0}^2}) & v_0 v_0^t (1 - \frac{1+\beta}{c_{s0}^2}) \\ v_0 v_0^t (1 - \frac{1+\beta}{c_{s0}^2}) & g^{rr} + v_0^2 (1 - \frac{1+\beta}{c_{s0}^2}) \end{bmatrix} \quad (80)$$

3.2.2 Constant Height Disk Model

For constant height disk model $H \propto \frac{1}{r}$. H does not change when linear perturbations are introduced in the fluid velocity and density. Now using continuity equation (77) and introducing linear perturbations, one gets after partially differentiating with t

$$(\partial_t F_1 \partial_t) \rho' + (\partial_t F_2 \partial_t) v' + (\partial_r F_3 \partial_t) \rho' + (\partial_r F_4 \partial_t) v' = 0 \quad (81)$$

where each F_i ($i \in \mathbb{N}, i = 1$ to 4) is function of r , the expressions are given below

$$\begin{aligned} F_1(r) &= \sqrt{-g} (H v_0^t + \alpha_2 \rho_0 H) \\ F_2(r) &= \sqrt{-g} \rho_0 H \alpha_1 \\ F_3(r) &= \sqrt{-g} H v_0 \\ F_4(r) &= \sqrt{-g} \rho_0 H \end{aligned}$$

Using equation (74), (75) and (81) we get

$$\partial_\mu (f^{\mu\nu}(r) \partial_\nu) \zeta' = 0 \quad (82)$$

where μ, ν indices run over t and r . 2×2 matrix, $f^{\mu\nu}$ is given by

$$f^{\mu\nu} = \frac{\sqrt{-g} H \rho_0}{h_0} \begin{bmatrix} -g^{tt} + (v_0^t)^2 (1 - \frac{1}{c_{s0}^2}) & v_0 v_0^t (1 - \frac{1}{c_{s0}^2}) \\ v_0 v_0^t (1 - \frac{1}{c_{s0}^2}) & g^{rr} + v_0^2 (1 - \frac{1}{c_{s0}^2}) \end{bmatrix} \quad (83)$$

3.2.3 Conical Disk Model

For conical disk model, H is a constant number. H does not change when linear perturbations are introduced in the fluid velocity and density because H does not depend on those quantities. Now using continuity equation (77) and introducing linear perturbations, one gets after partially differentiating with t

$$(\partial_t F_1 \partial_t) \rho' + (\partial_t F_2 \partial_t) v' + (\partial_r F_3 \partial_t) \rho' + (\partial_r F_4 \partial_t) v' = 0 \quad (84)$$

where each F_i ($i \in \mathbb{N}, i = 1$ to 4) is function of r , the expressions are given below

$$\begin{aligned} F_1(r) &= \sqrt{-g}(H v_0^t + \alpha_2 \rho_0 H) \\ F_2(r) &= \sqrt{-g} \rho_0 H \alpha_1 \\ F_3(r) &= \sqrt{-g} H v_0 \\ F_4(r) &= \sqrt{-g} \rho_0 H \end{aligned}$$

Using equation (74), (75) and (84) we get

$$\partial_\mu (f^{\mu\nu}(r) \partial_\nu) \zeta' = 0 \quad (85)$$

where μ, ν indices run over t and r . 2×2 matrix, $f^{\mu\nu}$ is given by

$$f^{\mu\nu} = \frac{\sqrt{-g} H \rho_0}{h_0} \begin{bmatrix} -g^{tt} + (v_0^t)^2 (1 - \frac{1}{c_{s0}^2}) & v_0 v_0^t (1 - \frac{1}{c_{s0}^2}) \\ v_0 v_0^t (1 - \frac{1}{c_{s0}^2}) & g^{rr} + v_0^2 (1 - \frac{1}{c_{s0}^2}) \end{bmatrix} \quad (86)$$

One trivial observation is that for constant height disk model and conical disk model as the disk height is not disturbed due to linear perturbations in the fluid, hence putting β to be zero in the matrix (80) one can obtain $f^{\mu\nu}$ for these models.

4 Concluding Remarks

Linear perturbation of several quantities obey massless scalar field equation in acoustic space time background. Unruh[1] first shown that linear perturbation of velocity potential obeys massless scalar field equation in curved space time background. In these papers[8, 9], it is explicitly shown that linear perturbation of mass accretion rate also gives acoustic metric. In our paper, we've shown that Bernoulli's constant also produces analogue gravity. The acoustic metric corresponding to the linear perturbation of the velocity potential and Bernoulli's constant are exactly the same but it is different while obtained by linear perturbing the mass accretion rate, which differs by a conformal factor. For a steady flow, Bernoulli's constant is obtained as the integral solution of Euler's equation. The velocity potential comes into the picture due to irrotationality condition and this condition is used in deriving the Euler's equation, hence linear perturbation of velocity potential gives exactly same acoustic metric as Bernoulli's constant.

For non-general relativistic background flow of adiabatic fluid, the Bernoulli's constant can be expressed as an additive term of various energy contribution to the total energy of the system. If one is interested to learn how the various sources of energy of the system, i.e; gravitational, mechanical, thermal and rotational, gets perturbed individually, the perturbation scheme of the Bernoulli's constant will be of great help to understand such physics and related issues.

Another advantage of constants of motion, ζ and f , over the velocity potential ψ , is that one can construct infinitely many quantities with any of constants of motion, ζ or f , whose linear perturbation obeys massless scalar field equation in curved space time. We introduce here an important finding of related interest. Linear perturbation of any algebraic function of ζ obeys massless scalar field equation in acoustic space time if the function and its first derivative with respect to ζ exist at the background value, i.e; at ζ_0 . The wave equation satisfied by linear perturbation of the function is exactly the same as the wave equation obeyed by linear perturbation of ζ . Same argument holds for the linear perturbation of mass accretion rate as well. To prove this, let's consider an algebraic function of ζ , $F(\zeta)$. We have

$$\partial_\mu f^{\mu\nu}(\vec{x}) \partial_\nu \zeta' = 0 \quad (87)$$

where ζ is perturbed linearly as

$$\begin{aligned}
\zeta(\vec{x}, t) &= \zeta_0(\vec{x}) + \zeta'(\vec{x}, t) \\
\Rightarrow F(\zeta(\vec{x}, t)) &= F(\zeta_0(\vec{x}) + \zeta'(\vec{x}, t)) \\
&= F(\zeta_0 + \zeta'(\vec{x}, t)) \\
&= F(\zeta_0) + \left(\frac{dF}{d\zeta} \right)_{\zeta_0} \zeta'(\vec{x}, t) \\
&= F_0 + F'
\end{aligned}$$

$\zeta_0, \left(\frac{dF}{d\zeta} \right)_{\zeta_0}$ are constant numbers because $F(\zeta)$ and its first derivative exist at ζ_0 and ζ_0 is a constant of motion. Linear perturbation of $F(\zeta)$ is a constant multiple of ζ' . Hence linear perturbation of $F(\zeta)$ obeys exactly same massless scalar field equation as obeyed by ζ' .

Similar argument holds for the case of mass accretion rate as well. In this fashion one can construct two disjoint sets of algebraic functions from two independent constants of motion ζ and f respectively. At this point, we thus argue that the linear perturbation of any quantity of fluid motion if obeys a massless scalar field equation, that equation will be same as wave equation satisfied by either of ζ or f .

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